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Office Hour: Send me an email first, then we will arrange a meeting (if you need it).
Tutorial Arrangement:

- (1330 - 1355/ 15:30 - 15:55): Problems.
- (1355 - 1415/ 15:55 - 16:15): Class exercises.
- (1415 - 1430/ 16:15 - 16:30): Submission of Class Exercise via Gradescope.
- (1430 - 1530/ 16:30 - 17:30): Late submission period.

1 Line Integrals of Vector Fields

1.1 Vector Fields

Vector fields: $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $\mathbf{F}(x_1, ..., x_n) = (F_1(x_1, ..., x_n), ..., F_m(x_1, ..., x_n))$. In this course you will mostly see vector fields defined in $\mathbb{R}^2$ or $\mathbb{R}^3$, then you can write it as

$$F = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$$

etc.

Example:

Gradient of a scalar function: Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a C^1 function, then

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is a vector field, and is called a gradient field.

1.2 Line Integrals

Let C be a curve with parametrization $\mathbf{r}(t)$ that gives C the counterclockwise direction. Then the line integral of $\mathbf{F}$ over C is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

Steps for calculation:

(i) Substitute $\mathbf{r}(t)$ into $\mathbf{F}$, i.e., find $\mathbf{F}(\mathbf{r}(t))$.

(ii) Differentiate $\mathbf{r}(t)$ and get $\mathbf{r}'(t)$.

(iii) Evaluate the integral with respect to t, where $t \in [a, b]$, i.e.,

$$\int_C \mathbf{F} \cdot dr = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

Exercise

Q1

Evaluate $\int_C \mathbf{F} \cdot dr$, where

$$\mathbf{F}(x, y, z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}$$

along the curve C given by

$$\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \sqrt{t}\mathbf{k},$$

for $t \in [0, 1]$

Solution:

$$\mathbf{F}(\mathbf{r}(t)) = (\sqrt{t}, t^3, -t^2)$$
$$\mathbf{r}'(t) = (2t, 1, \frac{1}{2\sqrt{t}})$$
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (\sqrt{t}, t^3, -t^2) \cdot (2t, 1, \frac{1}{2\sqrt{t}})\, dt = \int_0^1 2t\sqrt{t} + t^3 - \frac{t^2}{2\sqrt{t}}\, dt$$

1.2.1 Line Integral with respect to dx, dy, dz

Let $\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the following vector field

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$$

and C be the curve

$$\mathbf{r}(t) = r_1(t)\mathbf{i} + r_2(t)\mathbf{j} + r_3(t)\mathbf{k}$$

then

$$\int_C F_1(x, y, z)\, dx = \int_a^b F_1(\mathbf{r}(t))r_1'(t)\, dt,$$
$$\int_C F_2(x, y, z)\, dy = \int_a^b F_2(\mathbf{r}(t))r_2'(t)\, dt,$$
$$\int_C F_3(x, y, z)\, dz = \int_a^b F_3(\mathbf{r}(t))r_3'(t)\, dt.$$

To summarize,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (F_1, F_2, F_3) \cdot (dx, dy, dz) \\ &= \int_C F_1\, dx + \int_C F_2\, dy + \int_C F_3\, dz\end{aligned}$$

Exercise

Q2

Evaluate the line integral

$$\int_C -y\, dx + z\, dy + 2x\, dz,$$

where C is the helix $\mathbf{r}(t) = (\cos t, \sin t, t)$ for $0 \leq t \leq 2\pi$.

Solution:

$$\mathbf{F}(\mathbf{r}(t)) = (-\sin t, t, 2\cos t)$$
$$\mathbf{r}'(t) = (-\sin t, \cos t, 1)$$
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \sin^2 t + t\cos t + 2\cos t \, dt$$

1.3 Work Done

Exercise

Q3

Find the work done by the force field

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

in moving an object along the curve C parametrized by

$$\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + t^2\mathbf{j} + \sin(\pi t)\mathbf{k}$$

for $0 \leq t \leq 1$.

Solution:

$$\mathbf{F}(\mathbf{r}(t)) = (\cos(\pi t), t^2, \sin(\pi t))$$
$$\mathbf{r}'(t) = (-\pi\sin(\pi t), 2t, \pi\cos(\pi t))$$
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -\pi\cos(\pi t)\sin(\pi t) + 2t^3 + \pi\cos(\pi t)\sin(\pi t) \, dt = \int_0^1 2t^3 \, dt$$

1.4 Conservative Vector Fields

Important properties:
Let F be a continuous vector field in a connected, open set G in $\mathbb{R}^n$. The following statements are equivalent:

(i) $\mathbf{F}$ is independent of path;

(ii) For any closed curve C is G,
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0,$$

(iii) $\mathbf{F}$ is conservative.

If $\mathbf{F}$ is conservative, then it is the gradient of a potential function Φ, i.e., $\mathbf{F} = \nabla\Phi$. Then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla\Phi \cdot d\mathbf{r} \\ &= \int_a^b \nabla\Phi(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \Phi(\mathbf{r}(b)) - \Phi(\mathbf{r}(a))\end{aligned}$$

The proof above is shown in class, the idea is to use Chain rule in a "reverse" way.

Important property:
A vector field $\mathbf{F}$ is conservative if and only if

$$\frac{\partial F_k}{\partial x_j} = \frac{\partial F_j}{\partial x_k}$$

for all j, k. For example, if $\mathbf{F}(x_1, x_2, x_3) = (F_1(x_1, x_2, x_3), F_2(x_1, x_2, x_3), F_3(x_1, x_2, x_3))$, then $\mathbf{F}$ is conservative if and only if

$$\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}, \quad \frac{\partial F_1}{\partial x_3} = \frac{\partial F_3}{\partial x_1}, \quad \frac{\partial F_2}{\partial x_3} = \frac{\partial F_3}{\partial x_2}$$

Exercises

Q4

Find the potential function (if exists) for the vector field

$$\mathbf{F}(x, y, z) = (y^2 + 4zx)\mathbf{i} + y(2x + 2z)\mathbf{j} + (y^2 + 2x^2)\mathbf{k}.$$

Solution:
We have

$$\begin{aligned}\frac{\partial F_1}{\partial y} &= 2y = \frac{\partial F_2}{\partial x} \\ \frac{\partial F_1}{\partial z} &= 4x = \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial z} &= 2y = \frac{\partial F_3}{\partial y}\end{aligned}$$

hence $\mathbf{F} = \nabla\Phi$ for some Φ

$$\frac{\partial \Phi}{\partial x} = y^2 + 4zx \tag{1.1}$$

$$\frac{\partial \Phi}{\partial y} = y(2x + 2z) \tag{1.2}$$

$$\frac{\partial \Phi}{\partial z} = y^2 + 2x^2 \tag{1.3}$$

Integrating (1.1) with respect to x yields

$$\Phi(x, y, z) = \int y^2 + 4zx\, dx = xy^2 + 2zx^2 + g_1(y, z)$$

then differentiating the Φ above with respect to y yields

$$\frac{\partial \Phi}{\partial y} = 2xy + \frac{\partial g_1}{\partial y}$$

comparing with (1.2) yields

$$\frac{\partial g_1}{\partial y} = 2yz$$

implying

$$g_1(y,z) = zy^2 + g_2(z).$$

Now,

$$\Phi(x,y,z) = xy^2 + 2zx^2 + zy^2 + g_2(z).$$

Then

$$\frac{\partial \Phi}{\partial z} = 2x^2 + y^2 + \frac{\partial g_2}{\partial z}$$

comparing with (1.3) implies

$$\frac{\partial g_2}{\partial z} = 0 \implies g_2(z) = C$$

hence

$$\Phi(x,y,z) = xy^2 + 2zx^2 + zy^2 + C.$$